# Math3506, 2011. Model solutions

## Qu1

(a) This is a mutualism model. There are intraspecifc competition terms $-bx^2, -fy^2$, and also cooperative terms $cxy, exy$. The carrying capacity for $x$ is $a/b$ and for $y$ is $d/f$.

(b) Steady states are $(0,0)$, $(0,d/f)$, $(a/b,0)$ and any interior steady state solves

$$a - bx + cy = 0, \quad d + ex - fy = 0.$$

Solving we get

$$(x^*, y^*) = \frac{1}{bf - ce}(fa + cd, ea + bd),$$

so we require $bf > ce$ for an interior steady state to exist.

For stability, we find the stability matrix

$$M = \begin{pmatrix} (a - bx + cy) - bx & cx \\ ey & (d + ex - fy) - fy \end{pmatrix}.$$

At $(0,0)$ we have

$$M = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

which has eigenvalues $a, d$ which are both positive and hence $(0,0)$ is an unstable node. At the interior steady state $(x^*, y^*)$ we have

$$M = \begin{pmatrix} -bx^* & cx^* \\ ey^* & -fy^* \end{pmatrix}.$$

Thus $\lambda_1 + \lambda_2 = -bx^* - fy^* < 0$ and $\lambda_1\lambda_2 = x^*y^*(bf - ce) > 0$ when the interior steady state exists. Hence both eigenvalues have negative real part and the interior steady state is locally stable when it exists.

(c) (i) $x(0) = 0, y(0) > 0$. $x(t) = 0\ \forall t$ and $\dot{y} = y(d - fy)$ so $y(t)$ obeys the logistic equation and hence all $y(0) > 0$ go monotonically to $d/f$, i.e. $\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = d/f$. Similarly if (ii) $y(0) = 0$ then $y(t) = 0\ \forall t$, $\dot{x} = x(a - bx)$ and if $x(0) > 0$ then $x(t) \to a/b$ monotonically, i.e. $\lim_{t\to\infty} x(t) = a/b, \lim_{t\to\infty} y(t) = 0$.

(d) See fig 1

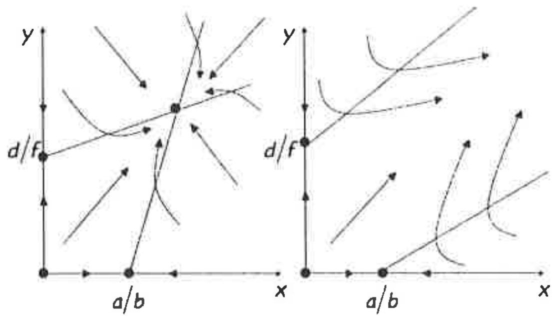



Figure 1: Qu 1 part . Left $bf > ce$, right $bf \leq ce$.

## Qu2

(a) Steady states are solutions of $N = rN(1 - N/K)$, so we have $N = 0$ and $N = K(1 - \frac{1}{r})$ which exists if $r > 1$.
Stability is determined by the eigenvalues $\lambda = f'(N^*)$ where $N^*$ is the steady state. But $f'(N) = r(1 - 2N/K)$. Hence $\lambda = f'(K(1 - \frac{1}{r})) = r(1 - 2(1 - \frac{1}{r})) = r(-1 + 2/r) = 2 - r$. We need $r > 1$ for the non-zero steady state to exist and $r < 3$ for $\lambda > -1$. Hence the positive steady state is linearly stable when $1 < r < 3$.

(b) See fig 2

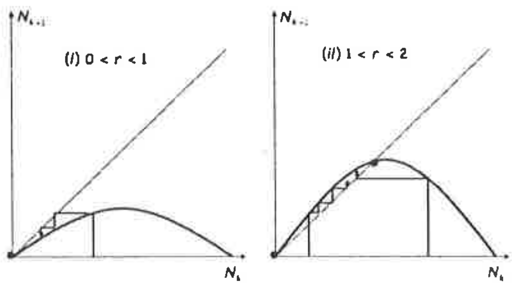



Figure 2: Qu2 part . Left $0 < r < 1$ and right $1 < r < 2$.

(c) A 2-cycle for $f$ is a pair of distinct points $N_-, N_+$ such that $f(N_-) = N_+$ and $f(N_+) = N_-$. We locate 2-cycles by solving $f^2(N) = N$ and discounting the steady states $N^*$, since although they satisfy $f^2(N^*) = N^*$, they are not part of a 2-cycle.

(d)

$$f^2(N) = r^2N(1-N)(rN^2 - rN + 1) = N.$$

To locate the 2-cycle we must throw away the 2 steady states $N^* = 0, 1 - 1/r$. Thus we first remove the root $N = 0$ and solve

$$r^2(1-N)(rN^2 - rN + 1) - 1 = 0.$$

Using that $r^2(1-N)(rN^2-rN+1)-1 = (r-rN-1)(1+r-(r+r^2)N+r^2N^2)$ we may discard the root $N = 1 - 1/r$ since this is a steady state. We are left with

$$0 = 1 + r - r(1+r)N + r^2N^2$$

so that $N_+N_- = (1+r)/r^2$ and $N_+ + N_- = (1+1/r) > 0$. Hence both roots $N_\pm$ have positive real part, and they are real if $[r(1+r)]^2 > 4r^2(1+r)$, i.e. $1 + r > 4$, which gives $r > 3$.

(e) The maximum of $f$ occurs at $N = K/2$ where $f(K/2) = rK/4 < K$ since $r < 4$. Hence if $N_k \in [0, K]$ then $N_{k+1} \in [0, K]$ and so the result follows by induction.

## Qu3

(a) This is a predator-prey model with $N$ the prey and $P$ the predator. The terms $-eN^2$ and $-fP^2$ are intraspecific competition. The term $-bNP$ models the effect on the prey of predation by $P$ and $dNP$ models the contribution of consumption of prey to the growth of the predator.

(b) The interior steady state satisfies

$$\begin{pmatrix} e & b \\ d & -f \end{pmatrix}\begin{pmatrix} N \\ P \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

Thus

$$\begin{pmatrix} N \\ P \end{pmatrix} = \frac{1}{fe+bd}\begin{pmatrix} f & b \\ d & -e \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{fa+bc}{fe+bd} \\ \dfrac{ad-ce}{fe+bd} \end{pmatrix}.$$

Thus the necessary and sufficient condition for a unique interior steady state is $ad > ce$.

(c) We put $D(t) = V(N(t), P(t))$. We may evaluate $D'(t)$ as follows:

$$\begin{aligned}
D'(t) &= \frac{d}{dt}V(N(t), P(t)) \\
&= \frac{d}{dt}\left(d(N(t) - N^* \log(N(t)/N^*)) + b(P(t) - P^* \log(P(t)/P^*))\right) \\
&= d\left(\dot{N}(t) - \frac{N^*}{N(t)}\dot{N}(t)\right) + b\left(\dot{P}(t) - \frac{P^*}{P(t)}\dot{P}(t)\right) \\
&= d\frac{\dot{N}(t)}{N(t)}(N(t) - N^*) + b\frac{\dot{P}(t)}{P(t)}(P(t) - P^*) \\
&= d(a - eN(t) - bP(t))(N(t) - N^*) + b(-c + dN(t) - fP(t))(P(t) - P^*)
\end{aligned}$$

Now, $a - eN^* - bP^* = 0, -c + dN^* - fP^*$ (since $(N^*, P^*)$ is an *interior* steady state). Hence

$$a - eN(t) - bP(t) = a - eN(t) - bP(t) - (a - eN^* - bP^*) = -e(N(t) - N^*) - b(P(t) - P^*),$$

and similarly $-c + dN(t) - fP(t) = d(N(t) - N^*) - f(P(t) - P^*)$. Hence we have

$$D'(t) = d[-e(N(t) - N^*) - b(P(t) - P^*)](N(t) - N^*) + b[d(N(t) - N^*) - f(P(t) - P^*)](P(t) - P^*).$$

Now let $X(t) = N(t) - N^*, Y(t) = P(t) - P^*$. Then

$$D'(t) = -deX(t)^2 - bfY(t)^2.$$

It is now clear that $D'(t) \leq 0$ with equality if and only if $X(t) = 0, Y(t) = 0$, i.e. $N(t) = N^*$, $P(t) = P^*$. Hence by the Lyapunov theorem any interior trajectory approaches the interior steady state as $t \to \infty$.

(d) When $e = 0$ and $f = 0$, $D'(t) = 0$ and $V(N(t), P(t)) =$ constant for all $t \geq 0$. The convex function $V$ has a unique minimum at the interior steady state $(c/d, a/b)$, and the level sets of $V$, which here correspond to the interior trajectories, are all closed curves. Hence the interior of the phase plane is a continuum of nested closed curves surrounding the interior steady state. (i.e. we get classic Lotka-Volterra Predator-Prey model).

## Qu4

(a) Set $u = N/K$, $\tau = \rho t$, $\alpha = \gamma P/K\rho$ and $\beta = A/K$.

(b) There is always the zero steady state $u = 0$. If $u$ is a non-zero steady state it must satisfy

$$1 - u - \frac{\alpha}{\beta + u} = 0,$$

that is

$$u^2 - (1 - \beta)u + \alpha - \beta = 0. \qquad (1)$$

When $\beta > \alpha$ the 2 roots of (1) are real with opposite sign, so there are 2 steady states in total, one of which is the origin.

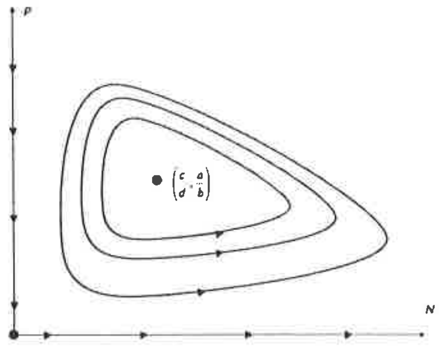
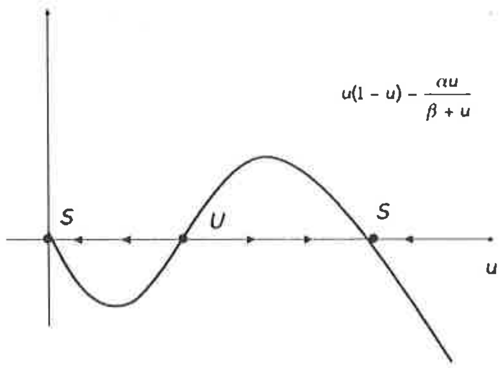



Figure 3: Qu3 part

(c) Let $\delta = (1-\beta)^2 - 4(\alpha - \beta) = (1+\beta)^2 - 4\alpha$. When $\delta > 0$ (1) has two real and distinct roots. If $\beta < 1$ then both these real roots are positive and we get 3 steady states which then gives the necessary and sufficient conditions for there to be exactly 3 distinct steady states to be $\alpha > \beta$, $\delta > 0$ and $\beta < 1$ which gives

$$1 > \beta > 2\sqrt{\alpha} - 1 \text{ and } \alpha > \beta.$$

Plotting $du/dt$ against $u$ shows that the origin is stable, as is the largest




steady state. The middle steady state is unstable.

(d) The value of $P$ means that initially $\alpha = \beta$, so that there are two zero roots and a positive root which is the value of the prey steady state. For

$P > \frac{K\rho}{4\gamma}\left(1+\frac{A}{K}\right)^2$ there is only one real root $u = 0$, which is globally stable and hence the prey population dies out.

## Qu5

(a) $\kappa$ is the per-capita rate of loss of immunity. $a$ is the per-capita rate of recovery of infectives. $r$ is the infection transmission rate.

(b) Adding the equations we find $N = S + I + R =$ constant. Thus we may set $S = N - I - R$ and eliminate $S$ to obtain

$$\begin{aligned}\dot{I} &= rI(N-I-R) - aI \\ \dot{R} &= aI - \kappa R\end{aligned}$$

(c) First note that $(I, R) = (0, 0)$ is a steady state. Any non-zero steady state is a solution to

$$\begin{aligned}0 &= r(N-I-R) - a \\ 0 &= aI - \kappa R\end{aligned}$$

So

$$(I, R) = \left(\frac{\kappa(rN-a)}{r(\kappa+a)}, \frac{a(rN-a)}{r(\kappa+a)}\right).$$

Thus we need $N = S_0 + I_0 > a/r$ for a positive steady state.

(d) See figure 4

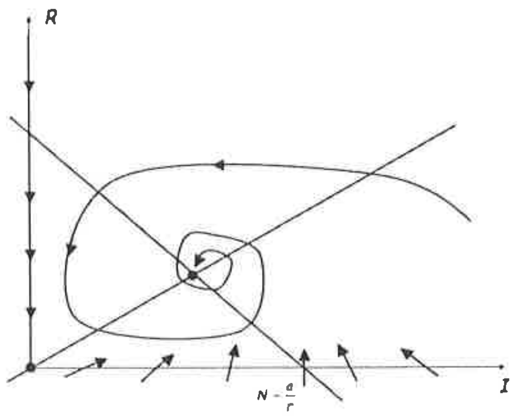



Figure 4: Phase plane for Qu5 part (d)

(e) Suppose that $S_0 + I_0 = N < \frac{a}{r}$. Then

$$\frac{\dot{I}}{I} = r(N - I - R) - a = (rN - a) - r(I + R).$$

Thus $\log I(t) = \log I_0 + (rN - a)t - r\int \{I(\tau) + R(\tau)\}\, d\tau$, which gives

$$I(t) = I_0 e^{(RN-a)t} e^{-r\int\{I(\tau)+R(\tau)\}\, d\tau}$$

Since $I(\tau), R(\tau) \geq 0$ we have $e^{-r\int\{I(\tau)+R(\tau)\}\, d\tau} \leq 1$. Since $N < a/r$ the term $e^{(RN-a)t} \to 0$ as $t \to \infty$. Hence $I(t) \to 0$ as $t \to \infty$.

(Careful arguments using a detailed phase plane diagram would also be acceptable.)